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Abstract

We compute spectra of symmetric random matrices defined on graphs exhibiting a modular structure. Modules are initially introduced as fully connected sub-units of a graph. By contrast, inter-module connectivity is taken to be incomplete. Two different types of inter-module connectivities are considered, one where the number of intermodule connections per-node diverges, and one where this number remains finite in the infinite module-size limit. In the first case, the results can be understood as a perturbation of a superposition of semicircular spectral densities one would obtain for uncoupled modules. In the second case, matters can be more involved, and depend in detail on inter-module connectivities. For suitable parameters we even find near-triangular-shaped spectral densities, similar to those observed in certain scale-free networks, in a system consisting of just two coupled modules. Analytic results are presented for the infinite module-size limit; they are well corroborated by numerical simulations.

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1. Introduction

Appreciation has steadily grown in recent years that theories of networked systems provide useful paradigms to understand complex processes in various branches of science and technology, including the evolution of the Internet and the World Wide Web, flows of information, power, or traffic, credit, market, or operational risk, food-webs in ecosystems, gene regulation, protein–protein interactions underlying metabolic processes or cell signaling, immune system response, information processing in neural networks, opinion formation, the adoption of new technologies in societies, or the spread of diseases or epidemics, and more (see, e.g. [1, 2] for recent reviews).

Random matrix theory has long been known to constitute a powerful tool to study topological properties of the graphs underlying networked systems [1, 3–6]. For example,

moments of the spectral density of an adjacency matrix describing a graph gives complete information about the number of walks returning to the originating vertex after a given number of steps.

Any form of heterogeneity existing in a graph, be it due to a scale-free distribution of connectivities [7], due to a small-world structure [8], or due to the existence of a modular structure in terms of identifiable sub-graphs [9], or a certain density of specific motifs existing in a graph would, therefore, manifest itself in the spectral density of states of the corresponding connectivity matrix [1, 3–6, 9]. A triangular density of states, for instance, has been hailed as a signature of a scale-free connectivity distribution emerging from network growth by preferential attachment [5].

Spectra of complex networks with heterogeneity arising from scale-free or small-world connectivity patterns were obtained through numerical diagonalization [5, 10]; analytical results for certain networks with power-law degree distributions exist, based either on the effective medium approximation (EMA) [11], or restricted to the limit of large average connectivity [12, 13]. A complete solution for general degree distributions with finite average connectivity (scale-free or other) has been obtained only recently, both for ensembles in the thermodynamic limit [14] and for large single instances [15], the latter generalized to non-Hermitean matrices in [16]. A highly efficient approximation for Poissonian random graphs at moderately large mean connectivity has appeared in [17]. We are not aware of studies other than numerical for the case of networks with small world [5] or explicitly modular structure [9].

In the present paper, we introduce a block-structured random matrix model to study spectral properties of coupled systems, or of a system with modules. The model can be thought of as a network composed of sub-networks, with weighted links within and between subnetworks. To simplify matters, we will initially consider the case where modules are defined as fully connected subnetworks. However, we note at the outset that this case is not substantially different from the one where intra-module connectivity is incomplete but with coordination that diverges in the infinite module-size limit. Inter-module connectivity is assumed to be incomplete, and two substantially different cases will be considered, one where the number of inter-module connections per node diverges, and one where this number remains finite in the infinite module-size limit.

We shall find that results for the first case can be rationalized in terms of perturbed superpositions of semi-circular spectral densities characteristic of the individual modules when considered on their own, whereas in the second case matters can be more involved, and depend in greater detail on inter-module connectivities. For suitable parameters we even find near-triangular-shaped spectral densities, similar to those observed in certain scale-free networks, already for a case of two coupled modules.

The remainder of the paper is organized as follows. In section 2, we present an elementary variant of a modular system, a system consisting of two coupled modules. This section mainly serves to introduce the formal definitions, and the method we will be using for computing the density of states, namely the replica method as proposed in the random matrix context by Edwards and Jones [18]. Details of the replica calculation for the extensively cross-connected case are given in section 2.1. The fundamentally different case of (two) finitely cross-connected modules is dealt with in section 2.2. In section 3, we present a generalization of the self-consistency equations derived earlier for the general M-modules case. Results for the extensively and finitely cross-connected two-modules case are presented and discussed in sections 4.1 and 4.2, respectively. The paper concludes with a summary and outlook in section 5.

2. A modular system

We begin by considering the most elementary modular system described by a $2N \times 2N$ symmetric block matrix of the form

$$H = \begin{pmatrix} M^{(1)} & V \\ V^t & M^{(2)} \end{pmatrix} \tag{1}$$

in which the symmetric $N \times N$ sub-matrices $M^{(1)}$, $M^{(2)}$ and the $N \times N$ coupling-sub-matrix V are random. We assume that elements of $M^{(1)}$ and $M^{(2)}$ are independently and identically Gaussian distributed: $M_{ij}^{(\mu)} \sim \mathcal{N}(0, J_\mu^2/N)$, $1 \leqslant i \leqslant j \leqslant N$, $\mu = 1, 2$, and that $V_{ij} = c_{ij} K_{ij}$ with $K_{ij} \sim \mathcal{N}(0, J_p^2/c)$, $1 \leqslant i, j \leqslant N$, while

$$p(c_{ij}) = \left(1 - \frac{c}{N}\right)\delta_{c_{ij},0} + \frac{c}{N}\delta_{c_{ij},1}. \tag{2}$$

Two different limits will be considered: (i) the so-called extensively cross-connected limit $c \to \infty$, $N \to \infty$, with the ratio c/N either remaining finite or approaching zero in the $N \to \infty$-limit[1] and (ii) the so-called finitely cross-connected limit, where c is kept constant, as $N \to \infty$. In this case, the distribution of inter-module connectivities is Poissonian in the thermodynamic limit, with average coordination c.

   We are interested in the spectral density of H,

$$\rho_N(\lambda) = \frac{1}{2N} \sum_{k=1}^{2N} \delta(\lambda - \lambda_k), \tag{3}$$

more precisely in its average $\overline{\rho_N(\lambda)}$ over the random matrix ensemble introduced above. Here, λ_k are the eigenvalues of H. We shall use $\overline{\rho(\lambda)}$ to denote the (average) spectral density in the thermodynamic limit,

$$\overline{\rho(\lambda)} = \lim_{N \to \infty} \overline{\rho_N(\lambda)}. \tag{4}$$

The spectral density is computed from the resolvent via

$$\rho_N(\lambda) = \lim_{\varepsilon \searrow 0} \frac{-2}{2N\pi} \operatorname{Im} \frac{\partial}{\partial \lambda} \ln \det \left[\lambda_\varepsilon \, \mathbb{I} - H\right]^{-1/2}, \tag{5}$$

in which $\lambda_\varepsilon \equiv \lambda - i\varepsilon$, and the inverse square root of the determinant is obtained as a Gaussian integral. Using $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ to denote N component vectors, and $\boldsymbol{u} = (\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)})$ to denote their concatenation, we get

$$\overline{\rho_N(\lambda)} = \lim_{\varepsilon \searrow 0} \frac{-2}{2N\pi} \operatorname{Im} \frac{\partial}{\partial \lambda} \left\langle \ln \left[\int \frac{\mathrm{d}\boldsymbol{u}^{(1)} \mathrm{d}\boldsymbol{u}^{(2)}}{(2\pi/i)^N} \exp\left\{ -\frac{i}{2} \boldsymbol{u} \cdot [\lambda_\varepsilon \, \mathbb{I} - H]\, \boldsymbol{u} \right\} \right] \right\rangle, \tag{6}$$

where angled brackets on the rhs denote an average over connectivities $\{c_{ij}\}$ and weights $\{M_{ij}^{(\mu)}\}$ and $\{K_{ij}\}$ of the non-vanishing matrix elements.

   The average of the logarithm is evaluated using replica

$$\overline{\rho_N(\lambda)} = \lim_{\varepsilon \searrow 0} \frac{-2}{2N\pi} \operatorname{Im} \frac{\partial}{\partial \lambda} \lim_{n \to 0} \frac{1}{n} \ln \langle Z_N^n \rangle, \tag{7}$$

with

$$Z_N^n = \int \prod_a \frac{\mathrm{d}\boldsymbol{u}_a^{(1)} \mathrm{d}\boldsymbol{u}_a^{(2)}}{(2\pi/i)^N} \exp\left\{ -\frac{i}{2} \sum_{a=1}^n \boldsymbol{u}_a \cdot [\lambda_\varepsilon \, \mathbb{I} - H]\, \boldsymbol{u}_a \right\}. \tag{8}$$

Here $a = 1, \ldots, n$ enumerates the replica.

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[1] The second alternative could more appropriately be referred to as sub-extensive, a distinction we are not going to make here for simplicity, as it does not affect the nature of results.

2.1. The extensively cross-connected case

The average $\langle Z_N^n \rangle$ is easily performed [18, 19]. We have

$$
\langle Z_N^n \rangle = \int \prod_a \frac{\mathrm{d}\boldsymbol{u}_a^{(1)} \mathrm{d}\boldsymbol{u}_a^{(2)}}{(2\pi/\mathrm{i})^N} \exp\left\{ -\frac{\mathrm{i}}{2} \sum_a \lambda_\varepsilon \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_a^{(1)} + \boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_a^{(2)} \right) \right\}
$$

$$
\times \left\langle \exp\left\{ \frac{\mathrm{i}}{2} \sum_a \boldsymbol{u}_a^{(1)} \cdot M^{(1)} \boldsymbol{u}_a^{(1)} \right\} \right\rangle \times \left\langle \exp\left\{ \frac{\mathrm{i}}{2} \sum_a \boldsymbol{u}_a^{(2)} \cdot M^{(2)} \boldsymbol{u}_a^{(2)} \right\} \right\rangle
$$

$$
\times \left\langle \exp\left\{ \mathrm{i} \sum_a \boldsymbol{u}_a^{(1)} \cdot V \boldsymbol{u}_a^{(2)} \right\} \right\rangle.
$$

Up to subdominant corrections from diagonal matrix elements this gives

$$
\begin{aligned}
\langle Z_N^n \rangle \simeq \quad & \int \prod_a \frac{\mathrm{d}\boldsymbol{u}_a^{(1)} \mathrm{d}\boldsymbol{u}_a^{(2)}}{(2\pi/\mathrm{i})^N} \exp\left\{ -\frac{\mathrm{i}}{2} \sum_a \lambda_\varepsilon \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_a^{(1)} + \boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_a^{(2)} \right) \right. \\
& \left. - \frac{J_1^2}{4N} \sum_{a,b} (\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_b^{(1)})^2 - \frac{J_2^2}{4N} \sum_{a,b} \left(\boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_b^{(2)} \right)^2 \right\} \\
& \times \prod_{ij} \left[1 + \frac{c}{N} \left(\exp\left\{ -\frac{J_p^2}{2c} \sum_{a,b} u_{ia}^{(1)} u_{ib}^{(1)} u_{ja}^{(2)} u_{jb}^{(2)} \right\} - 1 \right) \right].
\end{aligned}
\tag{9}
$$

Expanding the exponential in the product (for large c) and re-exponentiating one obtains

$$
\begin{aligned}
\langle Z_N^n \rangle \simeq \int \prod_a \frac{\mathrm{d}\boldsymbol{u}_a^{(1)} \mathrm{d}\boldsymbol{u}_a^{(2)}}{(2\pi/\mathrm{i})^N} \exp\left\{ -\frac{\mathrm{i}}{2} \sum_a \lambda_\varepsilon \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_a^{(1)} + \boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_a^{(2)} \right) - \frac{J_1^2}{4N} \sum_{a,b} \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_b^{(1)} \right)^2 \right. \\
\left. - \frac{J_2^2}{4N} \sum_{a,b} \left(\boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_b^{(2)} \right)^2 - \frac{J_p^2}{2N} \sum_{a,b} \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_b^{(1)} \right) \left(\boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_b^{(2)} \right) \right\}.
\end{aligned}
\tag{10}
$$

Decoupling of sites is achieved by introducing

$$
q_{ab}^{(\mu)} = \frac{1}{N} \sum_i u_{ia}^{(\mu)} u_{ib}^{(\mu)}, \quad \mu = 1, 2,
\tag{11}
$$

as order parameters and by enforcing their definition via δ-functions. This leads to

$$
\langle Z_N^n \rangle \simeq \int \prod_{\mu,a,b} \frac{\mathrm{d}q_{ab}^{(\mu)} \mathrm{d}\hat{q}_{ab}^{(\mu)}}{2\pi/N} \exp\{ N[G_1 + G_2 + G_3] \}
\tag{12}
$$

with

$$
G_1 = -\frac{J_1^2}{4} \sum_{ab} (q_{ab}^{(1)})^2 - \frac{J_2^2}{4} \sum_{ab} \left(q_{ab}^{(2)} \right)^2 - \frac{J_p^2}{2} \sum_{ab} q_{ab}^{(1)} q_{ab}^{(2)},
\tag{13}
$$

$$
G_2 = -\mathrm{i} \sum_{ab} \left(\hat{q}_{ab}^{(1)} q_{ab}^{(1)} + \hat{q}_{ab}^{(2)} q_{ab}^{(2)} \right),
\tag{14}
$$

4

$$G_3 = \ln\left[\int \prod_a \frac{\mathrm{d}u_a^{(1)}\mathrm{d}u_a^{(2)}}{2\pi/\mathrm{i}} \exp\left\{-\frac{\mathrm{i}}{2}\sum_{ab}\left(\lambda_\varepsilon \delta_{a,b} - 2\hat{q}_{ab}^{(1)}\right)u_a^{(1)}u_b^{(1)}\right.\right.$$

$$\left.\left. -\frac{\mathrm{i}}{2}\sum_{ab}\left(\lambda_\varepsilon \delta_{a,b} - 2\hat{q}_{ab}^{(2)}\right)u_a^{(2)}u_b^{(2)}\right\}\right]$$

$$= -\frac{1}{2}\ln\det\left(\lambda_\varepsilon \, \mathrm{I\!I} - 2\hat{q}^{(1)}\right) - \frac{1}{2}\ln\det\left(\lambda_\varepsilon \, \mathrm{I\!I} - 2\hat{q}^{(2)}\right). \tag{15}$$

2.1.1. Replica symmetry—fixed point equations. In the large N limit the density of states is dominated by the saddle-point contribution to equation (12). Adopting the by now well-established assumptions of replica symmetry, and invariance under rotation in the space of replica at the relevant saddle point [18]

$$q_{ab}^{(\mu)} = q_d^{(\mu)}\delta_{ab}, \qquad \hat{q}_{ab}^{(\mu)} = \hat{q}_d^{(\mu)}\delta_{ab}, \tag{16}$$

one has

$$G = G_1 + G_2 + G_3$$
$$\simeq n\left\{-\frac{J_1^2}{4}\left(q_d^{(1)}\right)^2 - \frac{J_2^2}{4}\left(q_d^{(2)}\right)^2 - \frac{J_p^2}{2}q_d^{(1)}q_d^{(2)}\right.$$
$$\left. -\mathrm{i}\hat{q}_d^{(1)}q_d^{(1)} - \mathrm{i}\hat{q}_d^{(2)}q_d^{(2)} - \frac{1}{2}\left[\ln\left(\lambda_\varepsilon - 2\hat{q}_d^{(1)}\right) + \ln\left(\lambda_\varepsilon - 2\hat{q}_d^{(2)}\right)\right]\right\} \tag{17}$$

where terms of order n^2 are omitted.

Stationarity of G requires that the RS order parameters solve the following fixed point equations:

$$-\mathrm{i}\hat{q}_d^{(1)} = \frac{1}{2}J_1^2 q_d^{(1)} + \frac{1}{2}J_p^2 q_d^{(2)}, \qquad -\mathrm{i}\hat{q}_d^{(2)} = \frac{1}{2}J_2^2 q_d^{(2)} + \frac{1}{2}J_p^2 q_d^{(1)},$$

as well as

$$q_d^{(1)} = \frac{1}{\mathrm{i}\lambda_\varepsilon - 2\mathrm{i}\hat{q}_d^{(1)}}, \qquad q_d^{(2)} = \frac{1}{\mathrm{i}\lambda_\varepsilon - 2\mathrm{i}\hat{q}_d^{(2)}}.$$

These can be combined by eliminating the conjugate variables to give

$$q_d^{(1)} = \frac{1}{\mathrm{i}\lambda_\varepsilon + J_1^2 q_d^{(1)} + J_p^2 q_d^{(2)}}, \qquad q_d^{(2)} = \frac{1}{\mathrm{i}\lambda_\varepsilon + J_2^2 q_d^{(2)} + J_p^2 q_d^{(1)}}. \tag{18}$$

2.1.2. Spectral density for the $c \to \infty$ case. The average spectral density is obtained by differentiation w.r.t. λ via equation (7). Only terms with *explicit* λ-dependence in G of equation (17) contribute at the saddle point. This gives

$$\overline{\rho(\lambda)} = \frac{1}{2\pi}\mathrm{Re}\left[q_d^{(1)} + q_d^{(2)}\right]. \tag{19}$$

For the limiting cases $J_p = 0$, describing two uncoupled systems, and $J_1 = J_2 = J$, describing a coupling of two systems with identical statistics of intra-system couplings we obtain the following explicit analytical results.

In the uncoupled case $J_p = 0$ the solution to equations (18) is

$$q_d^{(\mu)} = -\frac{\mathrm{i}\lambda_\varepsilon}{2J_\mu^2} \pm \frac{1}{2J_\mu^2}\sqrt{4J_\mu^2 - \lambda_\varepsilon^2} \tag{20}$$

giving

$$\overline{\rho(\lambda)} = \frac{1}{2\pi} \left[\frac{\Theta(4J_1^2 - \lambda^2)}{2J_1^2} \sqrt{4J_1^2 - \lambda^2} + \frac{\Theta(4J_2^2 - \lambda^2)}{2J_2^2} \sqrt{4J_2^2 - \lambda^2} \right], \quad (21)$$

i.e. a superposition of two independent Wigner semi-circular densities as expected.

In the case of two coupled systems which are statistically identical, with $J_1 = J_2 = J$, the solution to equations (18) is $q_d^{(1)} = q_d^{(2)} = q_d$ with

$$q_d = -\frac{i\lambda_\varepsilon}{2(J^2 + J_p^2)} \pm \frac{1}{2(J^2 + J_p^2)} \sqrt{4(J^2 + J_p^2) - \lambda_\varepsilon^2} \quad (22)$$

leading to

$$\overline{\rho(\lambda)} = \frac{\Theta(4(J^2 + J_p^2) - \lambda^2)}{2\pi(J^2 + J_p^2)} \sqrt{4(J^2 + J_p^2) - \lambda^2}, \quad (23)$$

i.e. a Wigner semi-circular density, with the radius of the semi-circle given by $r = 2\sqrt{J^2 + J_p^2}$.

In the asymmetric case $J_1 \neq J_2$ we solve the fixed point equations numerically. A non-zero coupling leads to a smoothing of the superposition of the two independent semi-circles as anticipated; see section 4.

2.2. Two finitely cross-connected modules

We now consider a system with the same basic setup, except that we assume the average number of cross-connections c to be finite (also in the thermodynamic limit). In this case, the analysis is considerably more involved. It combines techniques used in [18] for fully connected and of [19] for sparse random matrices, and more specifically the reformulation [14] of the sparse case that allows us to proceed to explicit results. In this case, the calculations can be carried out without restricting the distribution of the non-zero matrix elements of the inter-module connections to be Gaussian, and we will in developing the theory not make that restriction.

For the average of the replicated partition function we get

$$\langle Z_N^n \rangle \simeq \int \prod_a \frac{d\boldsymbol{u}_a^{(1)} d\boldsymbol{u}_a^{(2)}}{(2\pi/i)^N} \exp \left\{ -\frac{i}{2} \sum_a \lambda_\varepsilon \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_a^{(1)} + \boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_a^{(2)} \right) - \frac{J_1^2}{4N} \sum_{a,b} \left(\boldsymbol{u}_a^{(1)} \cdot \boldsymbol{u}_b^{(1)} \right)^2 \right.$$
$$\left. - \frac{J_2^2}{4N} \sum_{a,b} \left(\boldsymbol{u}_a^{(2)} \cdot \boldsymbol{u}_b^{(2)} \right)^2 + \frac{c}{N} \sum_{ij} \left(\left\langle \exp \left\{ iK \sum_a u_{ia}^{(1)} u_{ja}^{(2)} \right\} \right\rangle_K - 1 \right) \right\} \quad (24)$$

in analogy to equation (9), where $\langle \cdots \rangle_K$ represents an average over the K_{ij} distribution, which is as yet left open. As in equation (9), subdominant contributions coming from diagonal matrix elements are omitted.

Decoupling of sites is achieved by introducing order parameters

$$q_{ab}^{(\mu)} = \frac{1}{N} \sum_i u_{ia}^{(\mu)} u_{ib}^{(\mu)}, \quad \mu = 1, 2, \quad (25)$$

as in the extensively cross-connected case before, but in addition also the replicated densities

$$\rho^{(\mu)}(\boldsymbol{u}) = \frac{1}{N} \sum_i \prod_a \delta\left(u_a - u_{ia}^{(\mu)}\right), \qquad \mu = 1, 2, \tag{26}$$

as well as the corresponding conjugate quantities. This allows us to express equation (24) as an integral over the order parameters and their conjugates, combined with a functional integral over the replicated densities and their conjugates,

$$\langle Z_N^n \rangle = \int \prod_\mu \{\mathcal{D}\rho^{(\mu)} \mathcal{D}\hat{\rho}^{(\mu)}\} \int \prod_{\mu,a,b} \frac{\mathrm{d}q_{ab}^{(\mu)} \mathrm{d}\hat{q}_{ab}^{(\mu)}}{2\pi/N} \exp\{N[G_1 + G_2 + G_3]\}, \tag{27}$$

with

$$G_1 = -\frac{J_1^2}{4} \sum_{ab} \left(q_{ab}^{(1)}\right)^2 - \frac{J_2^2}{4} \sum_{ab} \left(q_{ab}^{(2)}\right)^2$$

$$+ c \int \mathrm{d}\rho^{(1)}(\boldsymbol{u}) \mathrm{d}\rho^{(2)}(\boldsymbol{v}) \left(\left\langle \exp\left\{\mathrm{i}K \sum_a u_a v_a\right\}\right\rangle_K - 1\right)$$

$$G_2 = -\mathrm{i} \sum_{ab} \left(\hat{q}_{ab}^{(1)} q_{ab}^{(1)} + \hat{q}_{ab}^{(2)} q_{ab}^{(2)}\right) - \mathrm{i} \int \mathrm{d}\boldsymbol{u} \left(\hat{\rho}^{(1)}(\boldsymbol{u}) \rho^{(1)}(\boldsymbol{u}) + \hat{\rho}^{(2)}(\boldsymbol{u}) \rho^{(2)}(\boldsymbol{u})\right)$$

$$G_3 = \ln \int \prod_a \mathrm{d}u_a \exp\left\{\mathrm{i}\hat{\rho}^{(1)}(\boldsymbol{u}) - \frac{\mathrm{i}}{2} \sum_{ab} \left(\lambda_\varepsilon \delta_{ab} - 2\hat{q}_{ab}^{(1)}\right) u_a u_b\right\}$$

$$+ \ln \int \prod_a \mathrm{d}u_a \exp\left\{\mathrm{i}\hat{\rho}^{(2)}(\boldsymbol{u}) - \frac{\mathrm{i}}{2} \sum_{ab} \left(\lambda_\varepsilon \delta_{ab} - 2\hat{q}_{ab}^{(2)}\right) u_a u_b\right\}.$$

Here we have introduced abbreviations of the form $\mathrm{d}\rho^{(\mu)}(\boldsymbol{u}) \equiv \mathrm{d}\boldsymbol{u}\, \rho^{(\mu)}(\boldsymbol{u})$ for integrals over densities where appropriate to simplify notation.

2.2.1. Replica symmetry and self-consistency equations. Equation (27) is evaluated by the saddle point method. The saddle point for this problem is expected to be replica symmetric and symmetric under rotation in the space of replica as in the extensively cross-connected case. In the present context, this translates to an ansatz of the form

$$q_{ab}^{(\mu)} = q_d^{(\mu)} \delta_{ab}, \qquad \hat{q}_{ab}^{(\mu)} = \hat{q}_d^{(\mu)} \delta_{ab}, \tag{28}$$

for the Edwards–Anderson-type order parameters [18] and

$$\rho^{(\mu)}(\boldsymbol{u}) = \int \mathrm{d}\pi^{(\mu)}(\omega) \prod_a \frac{\exp\left[-\frac{\omega}{2} u_a^2\right]}{Z(\omega)},$$

$$\mathrm{i}\hat{\rho}^{(\mu)}(\boldsymbol{u}) = \hat{c}^{(\mu)} \int \mathrm{d}\hat{\pi}^{(\mu)}(\hat{\omega}) \prod_a \frac{\exp\left[-\frac{\hat{\omega}}{2} u_a^2\right]}{Z(\hat{\omega})}, \tag{29}$$

i.e. an uncountably infinite superposition of complex Gaussians (with $\mathrm{Re}\,\omega \geqslant 0$ and $\mathrm{Re}\,\hat{\omega} \geqslant 0$) for the replicated densities and their conjugates [14]. Here we have introduced the shorthand

$$Z(\omega) = \int \mathrm{d}u \exp\left[-\frac{\omega}{2} u^2\right] = \sqrt{2\pi/\omega}. \tag{30}$$

$\hat{c}^{(\mu)}$ in the expressions for $\hat{\rho}^{(\mu)}$ are to be determined such that the densities $\hat{\pi}^{(\mu)}$ are normalized.

This ansatz translates path integrals over the replicated densities ρ and $\hat{\rho}$ into path integrals over the densities π and $\hat{\pi}$, giving

$$\langle Z_N^n \rangle = \int \prod_\mu \{\mathcal{D}\pi^{(\mu)} \mathcal{D}\hat{\pi}^{(\mu)}\} \int \prod_\mu \frac{\mathrm{d}q_d^{(\mu)} \mathrm{d}\hat{q}_d^{(\mu)}}{2\pi/N} \exp\{N[G_1 + G_2 + G_3]\} \quad (31)$$

with now

$$G_1 \simeq n\left[-\frac{J_1^2}{4}(q_d^{(1)})^2 - \frac{J_2^2}{4}(q_d^{(2)})^2 \right. $$
$$\left. + c \int \mathrm{d}\pi^{(1)}(\omega)\mathrm{d}\pi^{(2)}(\omega') \left\langle \ln \frac{Z_2(\omega, \omega', K)}{Z(\omega)Z(\omega')} \right\rangle_K \right], \quad (32)$$

$$G_2 \simeq -n\left[\mathrm{i}\hat{q}_d^{(1)} q_d^{(1)} + \mathrm{i}\hat{q}_d^{(2)} q_d^{(2)} \right]$$
$$- \sum_{\mu=1}^2 \left[\hat{c}^{(\mu)} + n\hat{c}^{(\mu)} \int \mathrm{d}\hat{\pi}^{(\mu)}(\hat{\omega})\mathrm{d}\pi^{(\mu)}(\omega) \ln \frac{Z(\hat{\omega} + \omega)}{Z(\hat{\omega})Z(\omega)} \right], \quad (33)$$

$$G_3 \simeq \sum_{\mu=1}^2 \left[\hat{c}^{(\mu)} + n \sum_{k=0}^\infty p_{\hat{c}^{(\mu)}}(k) \int \{\mathrm{d}\hat{\pi}^{(\mu)}\}_k \ln \frac{Z^{(\mu)}(\{\hat{\omega}\}_k)}{\prod_{\ell=1}^k Z(\hat{\omega}_\ell)} \right]. \quad (34)$$

Here $\{\mathrm{d}\hat{\pi}^{(\mu)}\}_k \equiv \prod_{\ell=1}^k \mathrm{d}\hat{\pi}^{(\mu)}(\hat{\omega}_\ell)$, while $\{\hat{\omega}\}_k \equiv \sum_{\ell=1}^k \hat{\omega}_\ell$, and

$$p_{\hat{c}^{(\mu)}}(k) = \frac{\hat{c}^{(\mu)k}}{k!} \exp[-\hat{c}^{(\mu)}] \quad (35)$$

is a Poissonian distribution with average $\langle k \rangle = \hat{c}^{(\mu)}$, and we have introduced the partition functions

$$Z^{(\mu)}(\{\hat{\omega}\}_k) = \int \frac{\mathrm{d}u}{\sqrt{2\pi/\mathrm{i}}} \exp\left[-\frac{1}{2}\left(\mathrm{i}\lambda_\varepsilon - 2\mathrm{i}\hat{q}_d^{(\mu)} + \{\hat{\omega}\}_k \right)u^2 \right]$$
$$= \left(\frac{\mathrm{i}}{\mathrm{i}\lambda_\varepsilon - 2\mathrm{i}\hat{q}_d^{(\mu)} + \{\hat{\omega}\}_k} \right)^{1/2}, \quad (36)$$

$$Z_2(\omega, \omega', K) = \int \mathrm{d}u\mathrm{d}v \exp\left[-\frac{1}{2}\left(\omega u^2 + \omega'v^2 - 2\mathrm{i}Kuv \right) \right] = \frac{2\pi}{\sqrt{\omega\omega' + K^2}}. \quad (37)$$

The stationarity conditions for $\pi^{(1)}(\omega)$ and $\pi^{(2)}(\omega)$ then read

$$\hat{c}^{(1)} \int \mathrm{d}\hat{\pi}^{(1)}(\hat{\omega}) \ln \frac{Z(\hat{\omega} + \omega)}{Z(\hat{\omega})Z(\omega)} = c \int \mathrm{d}\pi^{(2)}(\omega') \left\langle \ln \frac{Z_2(\omega, \omega', K)}{Z(\omega)Z(\omega')} \right\rangle_K + \mu_1 \quad (38)$$

and

$$\hat{c}^{(2)} \int \mathrm{d}\hat{\pi}^{(2)}(\hat{\omega}) \ln \frac{Z(\hat{\omega} + \omega)}{Z(\hat{\omega})Z(\omega)} = c \int \mathrm{d}\pi^{(1)}(\omega') \left\langle \ln \frac{Z_2(\omega', \omega, K)}{Z(\omega)Z(\omega')} \right\rangle_K + \mu_2, \quad (39)$$

with μ_1 and μ_2 being Lagrange multipliers to take the normalization of $\pi^{(1)}$ and $\pi^{(2)}$ into account. The stationarity conditions for $\hat{\pi}^{(\mu)}(\hat{\omega})$ are

$$c \int \mathrm{d}\pi^{(\mu)}(\omega) \ln \frac{Z(\hat{\omega} + \omega)}{Z(\hat{\omega})Z(\omega)} = \sum_{k \geqslant 1} k p_{\hat{c}^{(\mu)}}(k) \int \{\mathrm{d}\hat{\pi}^{(\mu)}\}_{k-1} \ln \frac{Z^{(\mu)}(\hat{\omega} + \{\hat{\omega}\}_{k-1})}{Z(\hat{\omega})\prod_{\ell=1}^{k-1} Z(\hat{\omega}_\ell)} + \sigma_\mu, \quad (40)$$

where σ_μ are once more Lagrange multipliers to take the normalization of $\hat{\pi}^{(\mu)}(\hat{\omega})$ into account.

The stationarity conditions for $q_d^{(\mu)}$ are

$$\mathrm{i}\hat{q}_d^{(\mu)} = -\frac{J_\mu^2}{2}q_d^{(\mu)}, \tag{41}$$

the corresponding ones for the conjugate variables give

$$
\begin{aligned}
q_d^{(\mu)} &= \sum_{k=0}^{\infty} p_{\hat{c}^{(\mu)}}(k) \int \{\mathrm{d}\hat{\pi}^{(\mu)}\}_k \left\langle u^2 \right\rangle_{\{\hat{\omega}\}_k}^{(\mu)} \\
&= \sum_{k=0}^{\infty} p_{\hat{c}^{(\mu)}}(k) \int \{\mathrm{d}\hat{\pi}^{(\mu)}\}_k \frac{1}{\mathrm{i}\,\lambda_\varepsilon + J_\mu^2 q_d^{(\mu)} + \{\hat{\omega}\}_k}
\end{aligned} \tag{42}
$$

in which $\langle\cdots\rangle_{\{\hat{\omega}\}_k}^{(\mu)}$ is defined as an average w.r.t. the Gaussian weight in terms of which $Z^{(\mu)}(\{\hat{\omega}\}_k)$ is defined. We have used equation (41) to express $\mathrm{i}\hat{q}_d^{(\mu)}$ in terms of $q_d^{(\mu)}$.

Following [20, 21], the stationarity conditions for $\pi^{(1)}(\omega)$ and $\pi^{(2)}(\omega)$ are rewritten in a form that suggests solving them via a population based algorithm. In the present case, we get [22, 14]

$$\hat{\pi}^{(1)}(\hat{\omega}) = \int \mathrm{d}\pi^{(2)}(\omega') \left\langle \delta\big(\hat{\omega} - \hat{\Omega}(\omega', K)\big)\right\rangle_K \tag{43}$$

and

$$\hat{\pi}^{(2)}(\hat{\omega}) = \int \mathrm{d}\pi^{(1)}(\omega') \left\langle \delta\big(\hat{\omega} - \hat{\Omega}(\omega', K)\big)\right\rangle_K, \tag{44}$$

in which

$$\hat{\Omega} = \frac{K^2}{\omega'}, \tag{45}$$

while

$$\pi^{(\mu)}(\omega) = \sum_{k\geqslant 1} \frac{k}{c} p_c(k) \int \{\mathrm{d}\hat{\pi}^{(\mu)}\}_{k-1} \delta\big(\omega - \Omega_{k-1}^{(\mu)}\big) \tag{46}$$

with

$$\Omega_{k-1}^{(\mu)} = \mathrm{i}\lambda_\varepsilon + J_\mu^2 q_d^{(\mu)} + \sum_{\ell=1}^{k-1} \hat{\omega}_\ell. \tag{47}$$

In equations (43), (44) and (46), we have already invested $\hat{c}^{(\mu)} = c$ to enforce that $\hat{\pi}^{(\mu)}$ are normalized.

For the spectral density we obtain the same formal result as for the extensively cross-connected system before,

$$\overline{\rho(\lambda)} = \frac{1}{2\pi}\mathrm{Re}\big[q_d^{(1)} + q_d^{(2)}\big]. \tag{48}$$

The solution of these coupled sets of equations is considerably more involved than for the extensively cross-connected systems considered earlier, as it involves solving equations for the coupled macroscopic order parameters $q_d^{(1)}$ and $q_d^{(2)}$, which are themselves expressed in terms of averages over self-consistent solutions of a pair of nonlinear integral equations parameterized by these order parameters. However, we have found that a population dynamics in which values of $q_d^{(1)}$ and $q_d^{(2)}$ are iteratively updated using equation (42) by sampling from the corresponding populations rapidly converges to the correct solution.

3. The multi-modular case

Finally, we consider systems with M modules, each of size N, mutually cross-connected with finite connectivity. Inside modules we may or may not have all-to-all Gaussian couplings of variances J_μ^2/N. In addition there are finitely many $\mathcal{O}(1)$ module-to-module couplings for each vertex, with average connectivities $c^{\mu\nu}$ for couplings between nodes in modules μ and ν (possibly including the case $\mu = \nu$). Evaluating this case requires a fairly straightforward generalization of the setup developed earlier. Here we only produce the fixed point equations and the final expression for the average spectral density. We get

$$\hat{\pi}^{(\mu)}(\hat{\omega}) = \sum_\nu \frac{c^{\mu\nu}}{c^\mu} \int d\pi^{(\nu)}(\omega) \left\langle \delta\big(\hat{\omega} - \hat{\Omega}(\omega, K)\big) \right\rangle_{\mu\nu} \tag{49}$$

and

$$\pi^{(\mu)}(\omega) = \sum_{k \geqslant 1} \frac{k}{c^\mu} p_{c^\mu}(k) \int \{d\hat{\pi}^{(\mu)}\}_{k-1} \delta\big(\omega - \Omega_{k-1}^{(\mu)}\big), \tag{50}$$

as well as

$$q_d^{(\mu)} = \sum_{k=0}^{\infty} p_{c^\mu}(k) \int \{d\hat{\pi}^{(\mu)}\}_k \frac{1}{\big(i\lambda_\varepsilon + J_\mu^2 q_d^{(\mu)} + \sum_{\ell=1}^{k} \hat{\omega}_\ell\big)} \tag{51}$$

with

$$\hat{\Omega}(\omega, K) = \frac{K^2}{\omega}, \tag{52}$$

and

$$\Omega_{k-1}^{(\mu)} = i\lambda_\varepsilon + J_\mu^2 q_d^{(\mu)} + \sum_{\ell=1}^{k-1} \hat{\omega}_\ell. \tag{53}$$

In equation (49), $\langle \cdots \rangle_{\mu\nu}$ denotes an average over the distribution of couplings connecting modules μ and ν, and we have the normalization $\sum_\nu c^{\mu\nu} = c^\mu$.

For the spectral density we obtain the (obvious) generalization of those obtained for the two modules case before, namely

$$\overline{\rho(\lambda)} = \frac{1}{M\pi} \text{Re} \sum_{\mu=1}^{M} q_d^{(\mu)}. \tag{54}$$

Generalizing these to the situation of varying module sizes $N_\mu = r_\mu N$, with $r_\mu > 0$, is straightforward.

4. Results

4.1. Extensively cross-connected systems

The results we obtain for extensively cross-connected systems can be understood in terms of superpositions of Wigner semi-circles one would expect for the spectral densities of the modules if they were uncoupled, but smoothed (and broadened) by the interaction. We have checked that our results agree perfectly with simulations, but have not included results of simulations in the figures for the extensively cross-connected systems below.

In figure 1, we explore the effect of the inter-module coupling strength J_p on the spectral density of a system of two coupled modules with intra-module coupling strengths J_μ differing
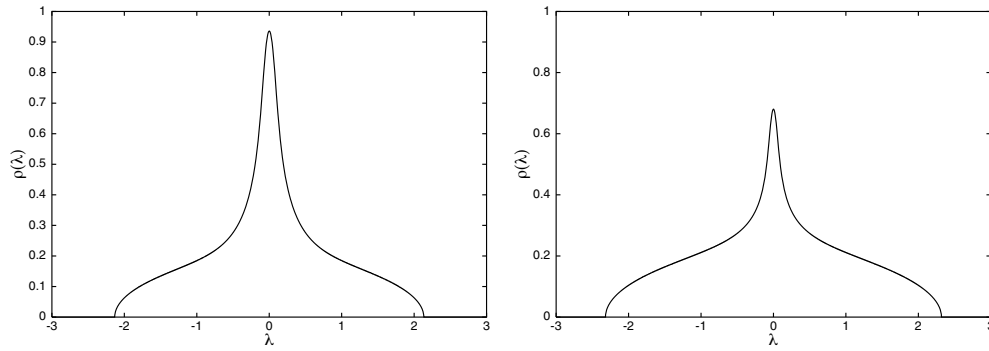
Figure 1. Spectral densities for $J_1 = 1$, $J_2 = 0.1$, and $J_p = 0.5$ (left), and $J_p = 0.75$ (right).
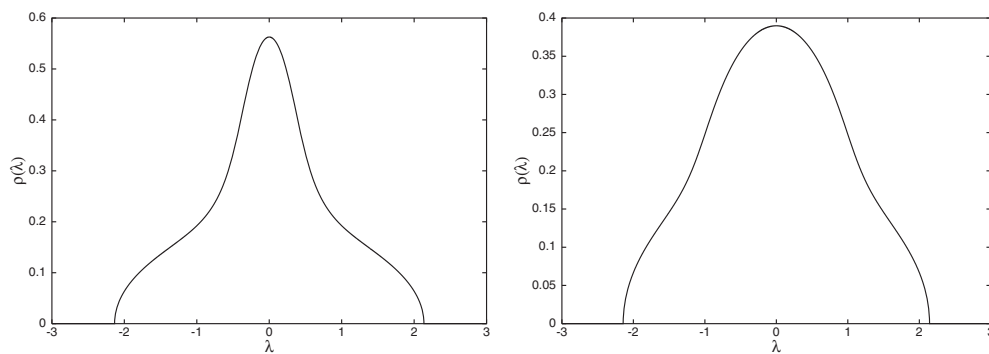

Figure 2. Spectral densities for $J_1 = 1$, $J_2 = 0.25$ (left), and $J_2 = 0.5$ (right), and $J_p = 0.5$ in both cases.

by one order of magnitude. The reader is invited to compare the results with those expected for in a situation where the two modules were non-interacting, namely a simple superposition of semi-circular densities of radii $2J_1$ and $2J_2$, respectively.

In figure 2, we keep the strength of the inter-module couplings but vary the ratio of the two intra-module couplings.

Results for uncoupled modules or for coupled modules with identical intra-module coupling statistics are not shown; they were found to be in perfect agreement with the simple analytical results presented in section 2.1.2.

4.2. Finitely cross-connected systems

The modifications generated by cross-connections that remain finite in number and strength for each node in each of the blocks are more pronounced than those created by extensive (infinitesimal) cross-connections. We mention just two fairly drastic modifications. First, the spectral density of a system of finitely cross-connected modules does no longer have sharp edges with square-root singularities of the spectral density at the edges as in the case of extensively cross-connected systems, where it derives from a perturbed superposition of semi-circular densities, but rather tails with decay laws that depend on the nature of the distribution of cross-connections. In the present case of Poisson distributions of cross-connections we find
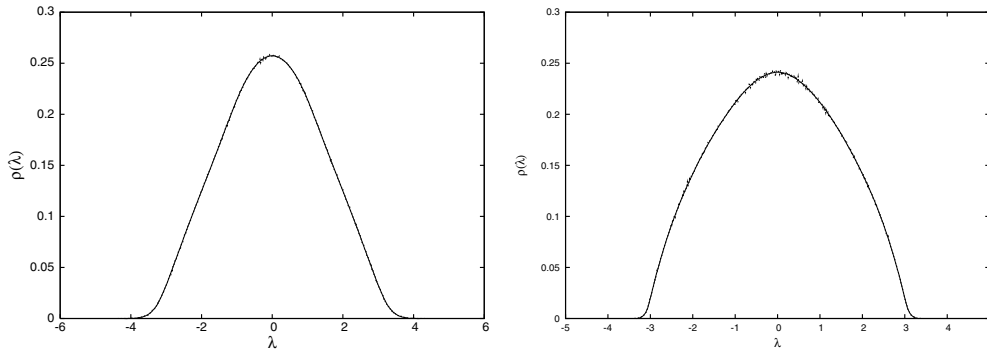
Figure 3. Spectral densities for $J_1 = J_2 = 1$ and $J_p = 1/\sqrt{c}$ for $c = 2$ (left) and $c = 5$ (right). Results of numerical diagonalizations of 500 matrices containing two coupled blocks, each of dimension 1000, are shown for comparison (dashed lines); they are virtually indistinguishable from results of the analytic theory.
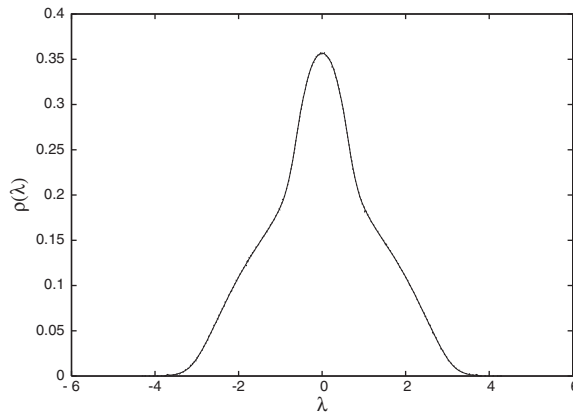

Figure 4. Spectral density for $J_1 = 1$, $J_2 = 0.5$ and $J_p = 1/\sqrt{c}$ for $c = 2$ (full line). Results of numerical diagonalizations of 500 matrices containing two coupled blocks, each of dimension 1000, are shown for comparison (dashed line) and are once more virtually indistinguishable from results of the analytic theory.

these tails to exhibit exponential decay. Second, even with identical intra-module coupling statistics the spectral density is no longer given by a simple semi-circular law as it is for the extensively cross-connected case, but rather exhibits marked deviations from the semi-circular law, with details depending both on the number and on the strengths of the cross-connections, as shown in figure 3.

As the numerics in the present finitely cross-connected case is considerably more involved we present results of the analytic theory together with checks against numerical diagonalization. Figures 3 and 4 demonstrate that the analytic results are in excellent agreement with numerical simulations, virtually indistinguishable for the parameters and the statistics used.

Figure 3 exhibits spectral densities of a two-module system with identical intra-module coupling statistics, and Gaussian cross-connections with a Poissonian degree statistics, of

average cross-coordinations 2 and 5, respectively. In the first case the spectral density has a shape close to triangular, though more rounded at the tip than what is known from spectral density of certain scale-free systems.

Figure 4 looks at a system of two modules having intra-module connections of different strengths and average cross-coordinations 2; results resemble the corresponding case of extensive cross-connectivity, apart from the exponential tails, which are an exclusive feature of the finitely cross-connected case.

5. Summary and conclusions

We have evaluated spectral densities of symmetric matrices describing modular systems. Modularity is regarded as one of several routes to create heterogeneity in interacting systems. In some biological systems in fact, modularity of interactions appears to be a natural consequence of compartmentalization; systems with cellular structure, or sub-structures within cells come to mind, where heterogeneity of interaction patterns due to modularity of the system would seem to enjoy a greater degree of plausibility than heterogeneity as observed in certain scale-free systems.

In any case, whenever large systems with different levels of organization are considered, modularity appears to be a feature to be reckoned with.

While the results for extensively cross-connected systems can be understood in terms of perturbed superpositions of semi-circular spectral densities characteristic of the individual modules if in isolation, the case of finitely cross-connected systems is more sensitive to details of the statistics of cross-connections.

We have evaluated examples only for systems containing two coupled modules, but presented the general theory for multi-modular systems in section 3. Our derivations and results were restricted to the case where entries of the connectivity matrix were sampled independently (leading to Poissonian degree distributions of inter-module connections in the finitely cross-connected case). Unlike in the case of a single module [22], the case where degree distributions of inter-module connections differ from Poissonian distributions requires more substantial modifications to the structure of the theory. We will deal with these, as well as with spectra of matrices corresponding to small-world systems [8] in a separate publication [23].

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